# Scaling Relations for Heating During Gliding Entry at Parabolic Speed

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## Nomenclature

$A$ = reference area of entry body
$C$ = constant, see Eq. (10)
$C_D$ = drag coefficient
$D$ = drag
$g$ = acceleration of gravity
$g_w$ = ratio of wall enthalpy to total enthalpy
$K$ = constant, see Eq. (7)
$L$ = lift
$m$ = vehicle mass
$q$ = heat transfer into the body per unit area
$R_0$ = planetary radius
$r_c$ = radius of curvature of flight path
$r_n$ = body nose radius
$t$ = time
$V$ = flight velocity
$V_s$ = surface grazing (parabolic) satellite speed 7.9 km/s
$x$ = distance measured along body surface
$y$ = flight altitude
$\beta$ = inverse scale height of atmosphere
$\gamma$ = flight-path angle below horizontal
$\rho$ = freestream density
$\phi$ = local body angle with respect to freestream

*Subscripts*

$i$ = initial value
$L,1$ = laminar boundary layer
max = maximum
$T,2$ = turbulent boundary layer
(*) = flight conditions corresponding to peak heating

## Introduction

ATMOSPHERIC entry from near-Earth satellite orbit using glide vehicles, such as the Space Shuttle, offers many advantages over the use of low-lift, high-drag capsules. Major advantages of glide vehicles are low deceleration loads and large cross range, affording a choice of conventional landing sites. To achieve the required lift, however, the glide vehicle needs a large lifting surface. Since the lifting surface is exposed to substantial aerodynamic heating, an extensive heat protection system is required. To maximize the vehicle payload, it is necessary to minimize the weight of the heat-protection system. In this Note, scaling relations are derived to show the influence of the primary vehicle parameters of ballistic coefficient $m/C_DA$ and lift-to-drag ratio $L/D$ on the peak heating rates and total heating, per unit area, for gliding entry at parabolic speed. Comparisons with stagnation point and windward centerline laminar and turbulent heating during three Space Shuttle flights are also presented.

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## Analysis

The equations of motion normal and parallel to the flight path are[1]

$$L - mg\cos\gamma = -mV^2/r_c \tag{1}$$

$$D - mg\sin\gamma = -m\frac{dV}{dt} \tag{2}$$

where

$$\frac{1}{r_c} = \frac{d\gamma}{ds} + \frac{\cos\gamma}{R_0 + y} \tag{3}$$

Assuming small flight-path angles $\cos\gamma \approx 1.0$, $\sin\gamma \approx \gamma$, and $d\gamma/ds \ll 1/(R_0+y)$, and noting that $y \ll R_0$, and therefore $g \doteq$ const, results in

$$L - mg \doteq -mV^2/R_0 \tag{4}$$

$$D - mg\gamma = -m\frac{dV}{dt} \tag{5}$$

During most of the entry, especially that part when heating is important

$$mg\gamma \ll D$$

and

$$D \doteq -m\frac{dV}{dt} \tag{6}$$

The flight-path expression for the equilibrium glide-path trajectory comes directly from Eq. (4) and is

$$\frac{\rho V^2}{V_s^2 - V^2} = \frac{2}{R_0}\frac{m}{C_DA}\frac{1}{L/D} = K \tag{7}$$

where the combination of terms on the right-hand side is considered constant and $V_s$ is the Earth surface-grazing satellite speed defined by $V_s = (g_0R_0)^{1/2}$. By combining Eqs. (4) and (6) with the assumption of an exponential atmosphere

$$\rho = \rho_0 e^{-\beta y} \tag{8}$$

the flight-path angle can be shown to be

$$\sin\gamma \approx \gamma = \frac{2}{\beta R_0}\frac{1}{L/D}\frac{V_s^2}{V^2} \tag{9}$$

For Earth, a vehicle with $L/D = 2$ flying at $V = 0.5V_s$ has a flight-path angle $\gamma = 0.25$ deg. Therefore, the small flight-path angle assumption is justified.

Now, it is assumed that the heating rate per unit area can be written in the form

$$\frac{dq}{dt} = C\rho^N V^M \tag{10}$$

where $N$, $M$, and $C$ are assumed constant. Equation (10) is a good approximation for both laminar[2] and turbulent[3] convection at a catalytic surface in the absence of boundary-layer mass addition. (Values of $N$, $M$, and $C$ are listed in the Appendix.)

The velocity and altitude of maximum heating can be found by substituting Eq. (7) into Eq. (10), setting the derivative of Eq. (10) with respect to $V$ equal to zero and solving, to yield

$$V_* = (M - 2N/M)^{1/2} V_s \tag{11}$$

$$\rho_* = \frac{2N}{M-2N}K = \left(\frac{2N}{M-2N}\right)\left(\frac{2}{R_0}\frac{m}{C_DA}\frac{1}{L/D}\right) \tag{12}$$

The peak heating rate per unit area is

$$\dot{q}_{\max} = C\left(\frac{4N}{R_0}\frac{m}{C_DA}\frac{1}{L/D}\right)^N\left(\frac{V_s}{\sqrt{M}}\right)^M(M-2N)^{M/2-N} \quad (13a)$$

or, in terms of the vehicle parameters only,

$$\dot{q}_{\max} \sim \left(\frac{m}{C_DA}\frac{1}{L/D}\right)^N \quad (13b)$$

The total heat transfer per unit area during the entry is

$$q = \int_0^t \frac{\mathrm{d}q}{\mathrm{d}t}\,\mathrm{d}t \quad (14)$$

By combining Eqs. (6), (7), and (10) and substituting the result into Eq. (14),

$$q = 2\frac{m}{C_DA}\int_V^{V_i} K^{N-1}C(V_s^2-V^2)^{N-1}V^{M-2N}\mathrm{d}V \quad (15)$$

where $V_i \leq V_s$. Assuming $C$ to be independent of velocity to a first order, then, for a laminar boundary layer ($N=0.5$ and $M=3$), Eq. (15) becomes

$$q_L = C_lK^{-1/2}V_s^2\frac{m}{C_DA}\left[\sin^{-1}\left(\frac{V_i}{V_s}\right)-\sin^{-1}\left(\frac{V}{V_s}\right) + \frac{V}{V_s}\sqrt{1-\left(\frac{V}{V_s}\right)^2}-\frac{V_i}{V_s}\sqrt{1-\left(\frac{V_i}{V_s}\right)^2}\right] \quad (16)$$

For a point on the body that experiences only laminar boundary-layer heating, and assuming entry occurs in the direction of the Earth's rotation (typical for most Shuttle flights), then Eq. (16) gives approximately

$$q_L \approx 0.9C_1K^{-1/2}V_s^2\frac{m}{C_DA} \quad (17a)$$

or

$$q_L \sim [(m/C_DA)(L/D)]^{0.5} \quad (17b)$$

When transition to a turbulent boundary layer occurs, the total heating integral in Eq. (15) becomes

$$\int_{V_T}^{V_i}\frac{V^{M-1}\mathrm{d}V}{(V_s^2-V^2)^{1/2}} + \int_0^{V_T}\frac{V^{M-1.6}\mathrm{d}V}{(V_s^2-V^2)^{1/5}} \quad (18)$$

where $V_T$ is the flight velocity at which the boundary layer has become turbulent. The velocity at which turbulent flow is fully established at a given location on the vehicle is a function of local Reynolds number, among other factors. (For example, heating was measured on the Space Shuttle at several locations on the windward centerline.[4] At a point about 48% of the vehicle length from the nose, fully turbulent boundary-layer heating occurred at $V=2.64$ km/s while, at the 30% point, transition was completed at about $V=2.28$ km/s.) The first integral in Eq. (18) is for the laminar heating contribution, and its solution (for $M=3$) is given by Eq. (16). The second integral in Eq. (18) represents the turbulent heating; while it cannot be evaluated exactly in closed form, an approximate analytic expression can be found that agrees within about 3% with numerical solutions, for cases where $V_T \leq V_s/2$, and is

$$\Delta q_T \approx C_2V_s^{M-1}K^{-0.2}\frac{m}{C_DA}\frac{(V_T/V_s)^{M-0.6}}{M-0.6} \quad (19a)$$

or

$$\Delta q_T \sim (m/C_DA)^{0.8}(L/D)^{0.2} \quad (19b)$$

The total heating at a body point that experiences boundary-layer transition at high speed is

$$q = \Delta q_L + \Delta q_T \quad (20)$$

where the laminar contribution can be determined from Eq. (16) with $V=V_T$. [If Eq. (16) is used, then $C_1$ (see Appendix) must be multiplied by 5.8 to compensate for $M=3$ being used, as opposed to $M=3.2$.]

## Comparison with Flight Data

Comparisons were made with the stagnation-point maximum heating rates determined from temperature measurements during Shuttle flights STS-5 and STS-41G.[5] According to Eq. (11), the maximum stagnation-point heating rate occurs at $V=0.82V_s$. During the entry of STS-5, peak stagnation-point heating occurred at $V=0.83V_s$, and for STS-41G it occurred at $V=0.85V_s$. The maximum stagnation point heating rates using Eq. (13a) were 50.6 W/cm$^2$ for STS-5 and 61.5 W/cm$^2$ for STS-41G. The flight-deduced values, assuming a fully catalytic surface and equilibrium flow, were 45.0 W/cm$^2$ for STS-5 and 54.5 W/cm$^2$ for STS-41G. The differences between Eq. (13a) and the data were 12% for STS-5 and 13% for STS-41G. Shuttle windward centerline heating data at flight speeds from 3.54 km/s to 2.28 km/s[4] were also compared with calculations using Eq. (10). (At these speeds, catalytic effects are neglegible.[5]) Computed laminar heating rates were 3–22% too low; the average was 18% low. Calculated turbulent heating rates were 10–27% too high; the average was 19% high. In view of the simplified nature of the present analysis, the agreement is considered good.

## Conclusions

In summary, the following conclusions are noted:

1) The total heat input per unit area is reduced by decreasing the vehicle's $m/C_DA$ and $L/D$. The reduction in total heat input with decreasing $m/C_DA$ is greater at a point on a body experiencing turbulent convection [see Eq. (19b)] than one having a laminar boundary layer [see Eq. (17b)]. However, $L/D$ affects total turbulent heating much less than total laminar heating.

2) For an area on a vehicle having only laminar flow, such as the stagnation point, the peak heating rate occurs at $V=0.82V_s$, Eq. (11), and a density altitude, Eq. (12), which is directly proportional to $K$. The corresponding peak heating rate is proportional to the square root of $K$ [see Eq. (13)]. Comparisons with maximum stagnation-point heating rates from two Space Shuttle flights agreed well with calculated values.

3) For an area on a vehicle having predominantly turbulent flow, the peak heating rate occurs at $V=0.75V_s$ and also at a density altitude [Eq. (12)] directly proportional to $K$. The maximum heating rate is proportional to $K^{0.8}$ [see Eq. (13)]. Naturally, the Reynolds number, etc., must be sufficiently high so that the boundary layer has become fully turbulent when the flight velocity has decreased to $0.75V_s$.

## Appendix

Values for the constants in Eq. (10) are listed below for a fully catalytic surface. The units of heating rate are W/cm$^2$ if velocity is in m/s and density in kg/m$^3$.

Stagnation point:

$$M=3 \qquad N=0.5 \qquad C=1.85(10^{-8})r_n^{-1/2}(1-g_w)$$

Laminar flat plate:

$$M=3.2 \qquad N=0.5$$

$$C_1 = 2.53(10^{-9})(\cos\phi)^{1/2}(\sin\phi)x^{-1/2}(1-g_w)$$

Turbulent flat plate:

$$N = 0.8 \quad V \leq 3962\ \text{m/s} \quad M = 3.37$$

$$C_2 = 3.35(10^{-8})(\cos\phi)^{1.78}(\sin\phi)^{1.6} x_T^{-1/5} \times (T_w/556)^{-1/4}(1 - 1.11 g_w)$$

$$N = 0.8 \quad V > 3962\ \text{m/s} \quad M = 3.7$$

$$C_2 = 2.20(10^{-9})(\cos\phi)^{2.08}(\sin\phi)^{1.6} x_T^{-1/5}(1 - 1.11 g_w)$$

### References

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[4]Throckmorton, D. A., "Benchmark Aerodynamic Heat Transfer Data from the First Flight of the Space Shuttle Orbiter," AIAA Paper 82-0003, Jan. 1982.

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# Evaluation of Emission Integrals for the Radiative Transport Equation

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## Introduction

ONE difficulty associated with discrete ordinates solutions to the radiative transport equation[1] is the accurate and computationally efficient evaluation of the emission integrals that appear in the particular solutions.[2-4] Experience has shown that most of the computational effort required in the determination of the radiative heat fluxes in participating media is due to time spent evaluating these emission integrals.[5] The purpose of this Note is to present and compare two efficient techniques for evaluating the emission integrals for situations involving arbitrary temperature profiles.

## Radiative Transport Equation

The general one-dimensional axisymmetric integrodifferential equation of radiative transfer in a plane parallel medium that absorbs, emits, and anisotropically scatters is given by[4]

$$\frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} = -\frac{I(\tau,\mu)}{\mu} + \frac{W}{2\mu}\int_{-1}^{1} I(\tau,\mu')\Phi(\mu,\mu')\mathrm{d}\mu' + \frac{(1-W)n^2 I_b(\tau)}{\mu} \tag{1}$$

Here, $\tau$ is the optical depth, $\mu$ the cosine of the polar angle $\theta$, $W$ the albedo of scattering, $I(\tau,\mu)$ the radiative intensity at depth $\tau$ and in the direction given by $\mu$, $n$ the refactive index, $I_b(\tau)$ Planck's blackbody intensity function, and $\Phi(\mu,\mu')$ the scattering phase function. The phase function can be represented by an $N$-term series of Legendre polynomials as

$$\Phi(\mu,\mu') = \sum_{\ell=0}^{N} A_\ell P_\ell(\mu) P_\ell(\mu') \tag{2}$$

where the coefficients $A_\ell$ can be determined either analytically or through the use of experimental data. If scattering within the medium is isotropic rather than anisotropic, the phase function takes on a value of unity.

Replacing the integral term in Eq. (1) with an $m$-order Gauss-Legendre quadrature yields the discrete ordinate approximation of the radiative transport equation, i.e.,

$$\frac{\mathrm{d}I(\tau,\mu_i)}{\mathrm{d}\tau} = -\frac{I(\tau,\mu_i)}{\mu_i} + \frac{W}{2\mu_i}\sum_{j=1}^{m} a_j I(\tau,\mu_j)\sum_{\ell=0}^{N} A_\ell P_\ell(\mu_i)P_\ell(\mu_j) + \frac{(1-W)n^2 I_b(\tau)}{\mu_i} \qquad i = 1,\ldots,m \tag{3}$$

where the $\mu_i$ and $\mu_j$ are the quadrature points, and the $a_j$ are the quadrature weights. Equation (3) represents a system of first-order, linear, inhomogeneous, ordinary differential equations, and solutions to this set of equations have been obtained for situations involving various levels of complexity.[1-4] The set of general solutions to Eq. (3) is given by[4]

$$\begin{aligned} I(\tau,\mu_i) = {} & \sum_{j=1}^{m/2}(1-\lambda_j\mu_j)\left[\frac{C_j e^{\lambda_j\tau} Y(\mu_i,\lambda_j)}{1+\mu_i\lambda_j} + \frac{C_{m+1-j}e^{-\lambda_j\tau}Y(\mu_i,-\lambda_j)}{1-\mu_i\lambda_j}\right] \\ & + (1-W)n^2 \sum_{j=1}^{m/2}\frac{K_{m+1-j}Y(\mu_i,-\lambda_j)}{1-\mu_i\lambda_j}\int_0^{\tau} I_b(t)e^{-\lambda_j(\tau-t)}\mathrm{d}t \\ & - (1-W)n^2 \sum_{j=1}^{m/2}\frac{K_j Y(\mu_i,\lambda_j)}{1+\mu_i\lambda_j}\int_{\tau}^{\tau_0} I_b(t)e^{-\lambda_j(t-\tau)}\mathrm{d}t \qquad i=1,\ldots,m \end{aligned} \tag{4}$$

where $\tau_0$ is the optical thickness of the medium, $C_j$ the $m$ constants of integration that must satisfy a given set of boundary conditions, and $\lambda_j$ the $m/2$ positive eigenvalues for the system of equations described by Eq. (3). The $Y$ functions are related to the phase function, and $K_j$ are a set of constants that must satisfy the system of equations given by

$$\sum_{j=1}^{m/2} K_j\left[\frac{Y(\mu_i,\lambda_j)}{1+\mu_i\lambda_j} - \frac{Y(\mu_i,-\lambda_j)}{1-\mu_i\lambda_j}\right] = \frac{1}{\mu_i}, \qquad i=1,\ldots,m/2 \tag{5}$$

$$K_{m+1-j} = -K_j, \qquad j = 1,\ldots,m/2 \tag{6}$$

The net radiative heat flux at depth $\tau$ can be obtained by integrating the intensity as follows:

$$Q_R(\tau) = 2\pi\int_{-1}^{1} I(\tau,\mu)\mu\,\mathrm{d}\mu \cong 2\pi\sum_{i=1}^{m} I(\tau,\mu_i)\mu_i a_i \tag{7}$$

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